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Exactly soluble random field Ising models in one dimension

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Abstract. We solve exactly the one-dimensional random field Ising model for two classes of magnetic field distributions: symmetric exponential (model I) and non-symmetric exponential (model II). For both models, expressions for the free energy at all finite temperatures are presented. The low-temperature region is examined in more detail; we obtain the zero-temperature energy and entropy in closed form; it is shown that the free energy of both models has an expansion in integer powers of temperature. Model I has a non-vanishing zero-point entropy for all values of the parameters as soon as randomness is diluted. In model II the zero-point entropy is zero except for a discrete sequence of values of one parameter. In some cases the zero-temperature magnetisation is positive whereas the average magnetic field is negative; the magnetisation may also change sign as a function of temperature.

1. Introduction

Magnetic models in random external fields have been the subject of considerable recent work. Their theoretical interest and difficulty is due to the combined effects of randomness and frustration, which are responsible for the existence of numerous degenerate or almost degenerate ground states, in close analogy to spin-glass models. The one-dimensional random field Ising model already exhibits interesting frustration effects, although physical quantities are singular only at zero temperature. After some earlier work (Fan and McCoy 1969, Azbel 1973, Lifshitz 1974), the interest in this problem was considerably revived by the introduction of the concept of frustration by Toulouse (1977). Exact solutions have been derived for some particular field distributions (Derrida *et al* 1978, Grinstein and Mukamel 1983). It has also been pointed out that the support of the distribution of local fields and related quantities is a Cantor set under some circumstances (Bruinsma and Aeppli 1983, Györgyi and Rujan 1984, Normand *et al* 1985). The regime where the ferromagnetic coupling J is much larger than the random fields has been examined by Derrida and Hilhorst (1983). However, up to now, the low-temperature behaviour is not known.

In this paper, we aim to present exact solutions of two particular classes of one-dimensional random field Ising models for all temperatures, using a method introduced by one of us (Nieuwenhuizen 1983, 1984a, b). It can be applied to a wide variety of one-dimensional disordered models and amounts to replacing the relevant integral equations by three-term recurrence relations with non-random coefficients. This method, if it works at all, assumes a very specific form of the distribution of the random interactions (usually exponential). An advantage is that also the diluted variant

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can be solved without much more effort (Nieuwenhuizen 1984b). The Green functions can also be computed (Nieuwenhuizen 1984a).

This method will allow us to solve exactly the two random field Ising models where the distributions of the magnetic fields h_i are as follows:

model I (symmetric exponential distribution)

$$h_i = H_r x_i \qquad (H_r > 0;\ -\infty < x_i < +\infty)$$

with

$$\rho(x_i) = (1-p)\delta(x_i) + \tfrac{1}{2}p\ \mathrm{e}^{-|x_i|} \tag{1.1}$$

model II (non-symmetric exponential distribution)

$$h_i = H_0 + H_r x_i \qquad (H_0 > 0;\ -\infty < H_r < +\infty,\ x_i > 0)$$

with

$$\rho(x_i) = (1-p)\delta(x_i) + p\ \mathrm{e}^{-x_i}. \tag{1.2}$$

The plan of the paper is as follows. In § 2 we present some general formalism. Section 3 is devoted to the solution of model I. Equation (3.14) gives an expression for the free energy at all temperatures. Equation (3.32) contains closed form expressions of the zero-point energy and entropy, and of the low-temperature behaviour of the specific heat. Model II is examined in § 4. The case $H_r > 0$ is elementary, since no frustration is present, and hence no interesting low-temperature behaviour. The result is given by equations (4.12)-(4.13). In the frustrated case ($H_r < 0$), we first study the large exchange coupling limit ($J \to \infty$). Our results agree with those of Derrida and Hilhorst (1983) and de Calan *et al* (1985), which predict the existence of a continuously varying exponent α^* in the singular part of the free energy; we also obtain the amplitude of that singularity in a closed form for arbitrary temperature. We then consider the low-temperature limit; our results for the zero-point energy and specific heat amplitude are presented in equations (4.39)-(4.41). A sequence of complex numbers α_j, containing the above mentioned exponent $\alpha^* = \alpha_0$, plays a central role in these expressions. Model II has a non-zero entropy at zero temperature for a discrete set of values of the steady part H_0 of the field, namely $H_0 = 2J/N$ ($N = 1, 2, 3, \ldots$). We finally give an efficient algorithm to compute the free energy for $H_r < 0$ at finite temperature (see equations (4.62)-(4.64)). Section 5 presents some concluding remarks.

2. Generalities

The Hamiltonian of an infinite Ising chain in a random magnetic field is

$$\mathscr{H} = -J \sum_i \sigma_i \sigma_{i+1} - \sum_i h_i \sigma_i \tag{2.1}$$

where the h_i are independent random variables with a common probability distribution $\rho(h)$. The partition function Z_N of a finite chain with N sites and periodic boundary conditions is

$$Z_N = \mathrm{Tr} \prod_{1 \le i \le N} T_i \qquad \text{with } T_i = \begin{pmatrix} \exp(\beta J + \beta h_i) & \exp(-\beta J - \beta h_i) \\ \exp(-\beta J + \beta h_i) & \exp(\beta J - \beta h_i) \end{pmatrix} \tag{2.2}$$

where $\beta = 1/T$ is the inverse temperature, and hence the free energy F of the model is given by

$$-\beta F = \lim_{N\to\infty} (1/N) \ln \operatorname{Tr} \prod_{1\leq i\leq N} T_i. \tag{2.3}$$

Let $(x_i; y_i)$ denote a sequence of two vectors such that $(x_0; y_0) = (1; 0)$ and $(x_{i+1}; y_{i+1})$ is the image by T_i of $(x_i; y_i)$. The ratios $R_i = x_i/y_i$ obey

$$R_{i+1} = \frac{\exp(2\beta J + 2\beta h_i) R_i + 1}{\exp(2\beta h_i) R_i + \exp(2\beta J)}. \tag{2.4}$$

When i becomes large, the distribution of R_i has a well-defined limit which is stationary, i.e. invariant under the substitution (2.4). A convenient way of dealing with this distribution is to introduce the following function of a complex variable u:

$$D(u) = \langle \ln(R-u)\rangle. \tag{2.5}$$

It can then be shown that the free energy F can be expressed as

$$-\beta F = -\beta J + \beta \bar{h} + \langle D[-\exp(2\beta J - 2\beta h)]\rangle \tag{2.6}$$

and that $D(u)$ obeys the following equation

$$D(u) = \left\langle D\left(\exp(-2\beta h)\frac{u\exp(2\beta J)-1}{\exp(2\beta J)-u}\right)\right\rangle + \ln[\exp(2\beta J) - u] + \beta F - \beta J + \beta \bar{h} \tag{2.7}$$

where $\bar{h}$ denotes the mean value of the random field h_i, and brackets stand for averages WRT the stationary distribution of R_i in equation (2.5) and WRT the magnetic field distribution in equations (2.6)-(2.7).

The function $D(u)$ is related to the distribution of the *local* fields $h_{i,\mathrm{loc}} = T\ln R_i$. The advantage of this function is twofold: its analytic structure is well suited for manipulations and analytic continuations, and it immediately gives the result for the quantity of interest, the free energy F. A very similar situation occurs in the study of random harmonic chains, where this approach was introduced by one of us (Nieuwenhuizen 1982). The quantities of interest in that problem are the inverse localisation length and the integrated density of states.

In the two models we consider in the present paper, the random fields are

$$h_i = H_0 + H_r x_i \tag{2.8}$$

where the x_i are dimensionless random variables, such that $\bar{x} = 0$ in model I and $\bar{x} = p$ in model II. It is then convenient to define a variable V through

$$V = \exp(-\beta J - \beta H_0)[\exp(2\beta J) - 1/R] \tag{2.9}$$

and the associated function

$$E(y) = \langle \ln(V-y)\rangle \tag{2.10}$$

which obeys the following equation:

$$E(y) = \langle E\{\exp(\beta J)[\exp(\beta H_0 + 2\beta H_r x) + \exp(-\beta H_0)] - 2y^{-1}\exp(2\beta H_r x)\sinh(2\beta J)\}\rangle$$
$$+ \ln y + \beta F - \beta H_r \bar{x}. \tag{2.11}$$

Equation (2.11) will be the starting point of our derivation of exact solutions in the following sections. We shall use throughout the paper the positive variables μ, ν, w defined through

$$\begin{aligned} &w=[2\sinh(2\beta J)]^{1/2} \\ &\cosh\mu=\exp(\beta J)\cosh(\beta H_0)w^{-1} \\ &\sinh\nu=\exp(2\beta J)\sinh(\beta H_0) \\ &\cosh\nu=\exp(\beta J)w\sinh\mu. \end{aligned} \tag{2.12}$$

In terms of these variables, in the pure (non-random) case ($p=0$), where each field assumes the value H_0, the stationary distribution of V is such that

$$E(y)=\ln(w\,\mathrm{e}^{-\mu}-y) \tag{2.13}$$

while the free energy is

$$\begin{aligned} -\beta F_{\text{pure}} &= \mu+\ln w \\ &= \beta J+\ln[\cosh\beta H_0+(\sinh^2\beta H_0+\exp(-4\beta J))^{1/2}]. \end{aligned} \tag{2.14}$$

Hereafter, we shall denote F_r the part of the free energy due to randomness

$$F_r=F-F_{\text{pure}} \tag{2.15}$$

and use the following usual definitions of internal energy, entropy, specific heat and magnetisation:

$$\begin{aligned} &U=\partial(\beta F)/\partial\beta \\ &S=\beta(U-F) \\ &C=-\beta^2\,\partial^2(\beta F)/\partial\beta^2=-\beta\,\partial S/\partial\beta \\ &M=-\partial F/\partial H_0. \end{aligned} \tag{2.16}$$

3. Model I: symmetric field distribution

3.1. Exact solution at finite temperature

In this section, we solve model I defined by $h_i=H_r x_i$ where the distribution of each x_i is

$$\rho(x)=\tfrac{1}{2}p\,\mathrm{e}^{-|x|}+(1-p)\delta(x). \tag{3.1}$$

Since H_0 vanishes, ν also vanishes, while μ is simply related to β through: $\exp(-2\mu)=\tanh(\beta J)$. Equation (2.11) then reads

$$\begin{aligned} &E(y)-\ln y-\beta F \\ &\quad=(1-p)E(\varphi(0,y))+\tfrac{1}{2}p\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}[E(\varphi(x,y))+E(\varphi(-x,y))] \end{aligned} \tag{3.2}$$

where the function φ, defined as

$$\varphi(x,y)=\exp(\beta J+2\beta H_r x)+\exp(\beta J)-y^{-1}\exp(2\beta H_r x)2\sinh(2\beta J) \tag{3.3}$$

satisfies the following identity:

$$\partial_x \varphi = 2\beta H_r y \left(\frac{y \exp(\beta J)}{2 \sinh(2\beta J)} - 1 \right) \partial_y \varphi. \tag{3.4}$$

Using this property of φ, we can integrate the RHS of (3.2) by parts twice and obtain

$$\left\{ 1 - \left[2\beta H_r \left(\frac{y^2 \exp(\beta J)}{2 \sinh(2\beta J)} - y \right) \partial y \right]^2 \right\} [E(y) - (1-p)E(y_1) - \ln y]$$
$$= pE(y_1) + \beta F \tag{3.5}$$

where $y_1 = 2\exp(\beta J) - 2y^{-1} \sinh(2\beta J)$.

Let us introduce now the variable

$$z = (w - y\,e^{-\mu})/(w - y\,e^{\mu}) \tag{3.6}$$

and the function

$$G(z) = E(y) + \ln(1 - z\,e^{2\mu}). \tag{3.7}$$

Equation (3.5) is then equivalent to

$$\{1 - \beta^2 H_r^2[(1-z^2)\partial_z]^2\}[G(z) - (1-p)G(z\,e^{-2\mu})]$$
$$= \beta F_r + pG(z\,e^{-2\mu}) + p\beta^2 H_r^2(1-z^2). \tag{3.8}$$

This differential-difference equation does not completely determine the function $G(z)$; we have to require that $G(z)$ has certain analyticity properties. From equation (2.4) we deduce that the support of the invariant distribution of R_i is the interval

$$\exp(-2\beta J) \leqslant R \leqslant \exp(2\beta J).$$

Hence the function $D(u)$ is analytic in a plane cut along this interval. The changes of variable (2.9) and (3.6) are such that the function $G(z)$ is analytic in a plane cut along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$. We can therefore expand $G(z)$ around $z = 0$

$$G(z) = G(0) - \sum_{k \geqslant 1} C_k z^k / k. \tag{3.9}$$

Moreover, $G(z)$ is an even function (all C_k with odd k vanish identically, due to the symmetry of the field distribution). It can also be easily shown that the coefficients C_k decrease to zero for large k, since they are moments of a random variable which lies between -1 and $+1$. Let

$$C_k = d_k p[1 - (1-p)\exp(-2k\mu)]^{-1}. \tag{3.10}$$

Equation (3.8) is then equivalent to the following recursion relation:

$$(k+1)d_{k+2} + (k-1)d_{k-2} = (2k + \varepsilon_k)d_k \tag{3.11}$$

where the ε_k are

$$\varepsilon_k = \frac{1}{k\beta^2 H_r^2} \frac{1 - \exp(-2k\mu)}{1 - (1-p)\exp(-2k\mu)} \tag{3.12}$$

together with the boundary conditions $d_0 = 1$ and $d_k \to 0$ as $k \to \infty$. The free energy is then simply related to d_2

$$F_r = -p\beta H_r^2(1 - d_2). \tag{3.13}$$

Equation (3.11) easily leads to the following continued fraction expansion of F_r:

$$F_r = -p\beta H_r^2\left(1 - \frac{1^2}{4+\varepsilon_{2-}}\,\frac{3^2}{8+\varepsilon_{4-}}\cdots\frac{(2n-1)^2}{4n+\varepsilon_{2n-}}\cdots\right) \tag{3.14}$$

which is rapidly convergent as long as the temperature is moderate. At low temperature, the number of terms needed to get a reasonable accuracy blows up as μ^{-1} (see below) and a new approach has to be used. Before going to that point, let us illustrate our result (3.14) and present in figure 1 plots of the specific heat against temperature for $H_r = 1.5$ and $J = 1$ and for different values of p. The low-temperature behaviour of $C(T)$ is linear in T as soon as p is non-zero (see equation (3.32c) below).

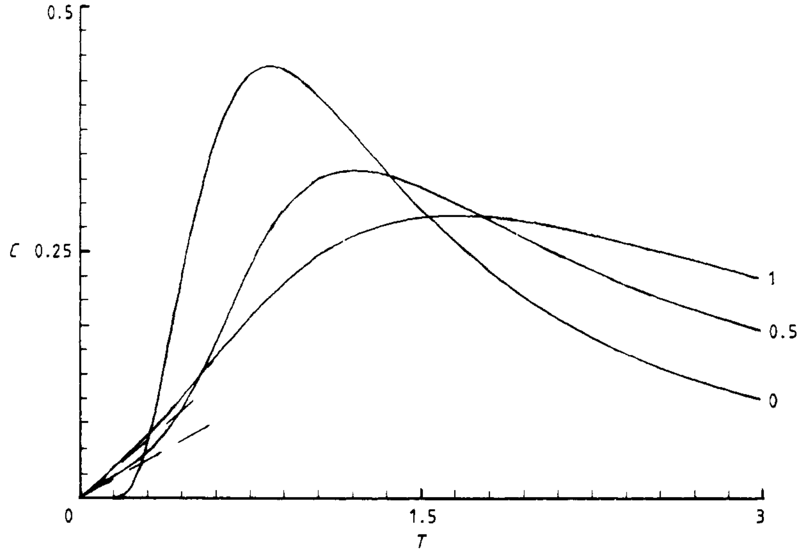

Figure 1. Plot of the specific heat against temperature for model I with $H_r = 1.5$ and $J = 1$. Values of p are indicated on the curves.

3.2. *Low-temperature behaviour*

We now focus our attention on the low-temperature region: $T = \beta^{-1} \ll J$. In this limit, the variable μ is exponentially small

$$\mu = \exp(-2\beta J) + \mathrm{O}(\exp(-6\beta J))$$

and the ε_k of equation (3.12) exhibit a crossover at $k \sim \mu^{-1} \sim \exp(2\beta J)$; namely $\varepsilon_k \sim 2\mu/p\beta^2 H_r^2$ for $k \ll \mu^{-1}$, while $\varepsilon_k \sim 1/k\beta^2 H_r^2$ for $k \gg \mu^{-1}$. In the first region ($k \ll \mu^{-1}$), ε_k is exponentially small in β; we shall neglect throughout the following such exponential contributions. The solution of equation (3.11) for $\varepsilon_k = 0$ reads (k even)

$$d_k = 1 + \frac{F_r}{p\beta H_r^2}\sum_{l=1}^{k/2}\frac{1}{2l-1}. \tag{3.15}$$

For $1 \ll k \ll \mu^{-1}$, d_k therefore behaves as

$$d_k = 1 + \frac{F_r}{2p\beta H_r^2}[\ln k + \ln 2 + \gamma + \mathrm{O}(k^{-1})] \tag{3.16}$$

where γ denotes Euler's constant.

For values of k comparable to μ^{-1}, we replace equation (3.11) by the following differential equation:

$$4(kd''+d')=\varepsilon d \tag{3.17}$$

where d and ε are now considered as functions of the real variable k.

The change of variable

$$y=(2\beta H_r)^{-1}\ln(2k\mu) \tag{3.18}$$

transforms equation (3.17) into the following Schrödinger-type equation:

$$d''(y)-u(y)d(y)=0 \tag{3.19}$$

where the 'potential' $u(y)$ is

$$u(y)=\frac{1-\exp[-\exp(\lambda y)]}{1-(1-p)\exp[-\exp(\lambda y)]} \tag{3.20}$$

with $\lambda=2\beta H_r$. The appropriate boundary condition is $d(y)\approx \mathrm{e}^{-y}$ as $y\to+\infty$. The solution then behaves as $d(y)\approx -Ay+B$ as $y\to-\infty$. If we now match this linear behaviour with equation (3.16), we obtain the free energy F as a simple function of the ratio B/A:

$$F=-J-pH_r^2(J+H_rB/A+\gamma T/2)^{-1}. \tag{3.21}$$

From this equation the large exchange coupling ($J\to\infty$) behaviour can be read immediately, since B/A does not depend on J. We shall come back to this point in § 4.

The coefficients A and B can be expanded as power series in T. A systematic way of dealing with these expansions is to introduce the Laplace transform of $d(y)$

$$D(z)=\int_{-\infty}^{+\infty}\mathrm{e}^{zy}\,\mathrm{d}(y)\,\mathrm{d}y \tag{3.22}$$

in terms of which equation (3.19) is equivalent to

$$D(z)=z^{-2}\int(\mathrm{d}s/2\pi\mathrm{i}s)\Gamma(1-s/\lambda)f(s/\lambda)D(z+s) \tag{3.23}$$

$$0<\mathrm{Re}\, s<\lambda \qquad 0<\mathrm{Re}(z+s)<1$$

where

$$f(s)=p\sum_{n\geqslant 0}(1-p)^n(n+1)^s. \tag{3.24}$$

In order to solve equation (3.23), let us iterate it by substituting in the integral the $T=0$ solution

$$D_0(z)=[z^2(1-z)]^{-1} \tag{3.25}$$

since $d_0(y)=\mathrm{e}^{-y}$ ($y\geqslant 0$); $d_0(y)=1-y$ ($y\leqslant 0$). We obtain

$$D_1(z)=[z^2(1-z)]^{-1}\Gamma[1-(1-z)/\lambda]f[(1-z)/\lambda]+\mathrm{O}(\lambda^{-4}). \tag{3.26}$$

Then, repeating the same procedure with $D_1(z)$,

$$D_2(z)=D_1(z)+\frac{1}{\lambda z^2}\sum_{n\geqslant 1}\frac{(-1)^n}{n!}\times\frac{f(n)\Gamma[n+1-(1-z)/\lambda]f[-n+(1-z)/\lambda]}{(z+n\lambda)^2(z-1+n\lambda)}+\mathrm{O}(\lambda^{-7}). \tag{3.27}$$

This sequence of approximants converges quickly: it can indeed be shown by induction that D_n and D_{n-1} differ only by terms of order λ^{-3n-1}. The coefficients A and B are such that

$$D(z) \underset{z\to 0}{=} A/z^2 + B/z + \text{regular part}. \tag{3.28}$$

We obtain A and B as power series in T by expanding $D(z)$ in powers of λ^{-1} and z. It is sufficient to use the approximant $D_1(z)$ to get A and B at order T^2

$$\begin{aligned} A &= 1 + \frac{\gamma + s_1}{2H_r} T + \frac{\gamma^2 + \pi^2/6 + 2\gamma s_1 + s_2}{8H_r^2} T^2 + \dots \\ B &= 1 - \frac{\gamma^2 + \pi^2/6 + 2\gamma s_1 + s_2}{8H_r^2} T^2 + \dots \end{aligned} \tag{3.29}$$

where the quantities s_k are

$$s_k = f^{(k)}(0) = p \sum_{n\geq 0} (1-p)^n [\ln(n+1)]^k. \tag{3.30}$$

By inserting equation (3.29) into equation (3.21), we obtain the final result

$$F = F_0 + F_1 T + F_2 T^2 + \dots \tag{3.31}$$

where

$$F_0 = U_0 = -J - (pH_r^2)/(J + H_r) \tag{3.32a}$$

$$-F_1 = S_0 = \frac{pH_r^2 s_1}{2(J+H_r)^2} \tag{3.32b}$$

$$-2F_2 = \Gamma_0 = \lim_{T\to 0} \frac{C}{T} = \frac{pH_r^2 s_1^2}{2(J+H_r)^3} + \frac{pH_r(s_2 - s_1^2 + \pi^2/6)}{2(J+H_r)^2}. \tag{3.32c}$$

Before discussing these expressions, let us mention that the present method can be pursued up to an arbitrary order T^l. The coefficients F_l are not functions of the s_k only. The lowest order at which other terms appear is T^4, where the sum in equation (3.27) generates a term proportional to aT^4 in F_r, with

$$a = \sum_{n\geq 1} (-1)^n f(n) f(-n) n^{-3}. \tag{3.33}$$

Let us now discuss briefly the physical contents of our results. The ground-state *energy* $U_0 = F_0$ has the very simple rational expression (3.32a). For $H_r \gg J$, the part due to randomness ($U_{0r} = U_0 + J$) behaves as $-pH_r$, since every spin which feels a non-zero field is aligned with it in that limit. The zero-point *entropy* $S_0 = -F_1$ is entirely due to field configurations where n successive sites have a zero field: $h_1 = \dots = h_n = 0$, while $h_0 > 0$ and $h_n < 0$ (or vice versa) are large enough. In this situation, the $(n+1)$ spin configurations where $\sigma_0 = \sigma_1 = \dots = \sigma_l = 1$; $\sigma_{l+1} = \dots = \sigma_{n+1} = -1$ (or vice versa) with $l = 0, \dots, n$ are degenerate. For $H_r \gg J$, the probability of these field configurations is just $p^2(1-p)^n/2$, and their entropy is $\ln(n+1)$. This fully agrees with equation (3.32b).

When the probability p is small, the zero-temperature entropy and specific heat amplitude have logarithmic singularities. Indeed, since the sum in equation (3.30) is dominated by large values of n in the $p\to 0$ limit, it is justified replacing it by an integral, which leads to

$$s_k = |\ln p|^k [1 + k\gamma/\ln p + \dots] \tag{3.34}$$

and hence

$$S_0 \sim \frac{H_r^2}{2(J+H_r)^2} p|\ln p| \qquad (p \to 0) \tag{3.35}$$

$$\Gamma_0 \sim \frac{H_r^2}{2(J+H_r)^3} p(\ln p)^2 \qquad (p \to 0). \tag{3.36}$$

In the other limit ($p \to 1$), the zero-temperature entropy vanishes linearly

$$S_0 \sim \frac{\ln 2}{2} \frac{H_r^2}{(J+H_r)^2} (1-p) \qquad (p \to 1) \tag{3.37}$$

while the specific heat amplitude remains finite

$$\Gamma_0 = \frac{\pi^2}{12} \frac{H_r}{(J+H_r)^2} \qquad (p = 1). \tag{3.38}$$

These results are illustrated in figures 2 and 3. Figure 2 shows a plot of the quantity $ps_1(p)$ which contains the p dependence of the zero-point entropy S_0 (see equation (3.32b)). Figure 3 shows plots of the specific heat amplitude Γ_0 against H_r for $J=1$ and different values of p.

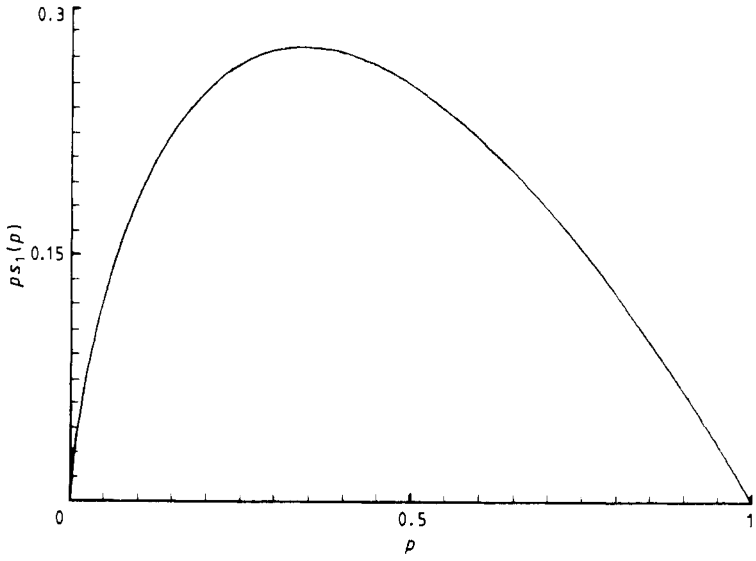

Figure 2. Plot of the quantity $ps_1(p)$ characterising the p dependence of the zero-point entropy of model I.

4. Model II: non-symmetric field distribution

4.1. *Preliminaries*

This section is devoted to the solution of model II where the random magnetic fields read: $h_i = H_0 + H_r x_i$, with $H_0 > 0$ and both possible signs for H_r, and where the distribution of each x_i reads

$$\rho(x) = p\,\mathrm{e}^{-x} + (1-p)\delta(x) \qquad (x \geq 0). \tag{4.1}$$

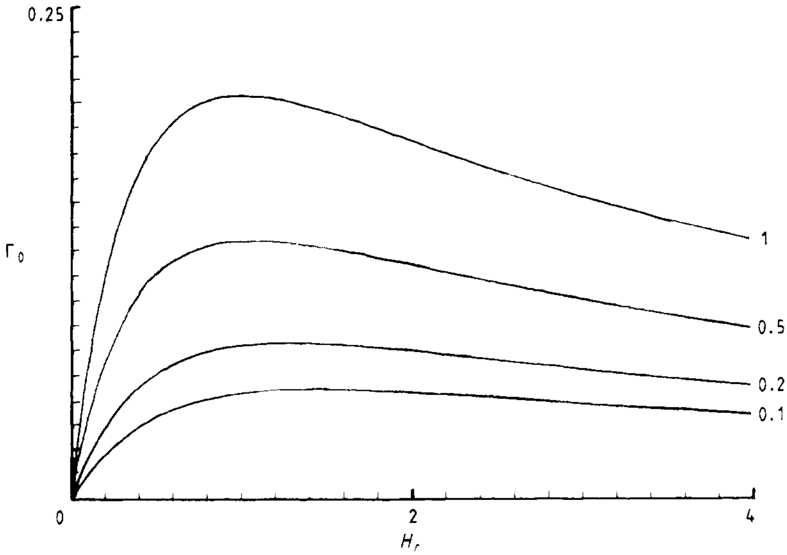

Figure 3. Plot of the specific heat amplitude Γ_0 against H_r for model I with $J=1$. Values of p are indicated on the curves.

The way we obtain an exact expression of the random part F_r of the free energy is very similar to the solution of model I. We start again from equation (2.11) which now reads

$$E(y)-\ln y-\beta F+p\beta H_r=(1-p)E(\varphi(0,y))+p\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}E(\varphi(x,y)) \tag{4.2}$$

where the function φ is defined as

$$\varphi(x,y)=\exp[\beta(J+H_0+2H_rx)]+\exp[\beta(J-H_0)]-y^{-1} \times\exp(2\beta H_rx)2\sinh(2\beta J) \tag{4.3}$$

and satisfies the following identity:

$$\partial_x\varphi=2\beta H_ry\left\{\frac{y\exp[\beta(J+H_0)]}{2\sinh(2\beta J)}-1\right\}\partial_y\varphi. \tag{4.4}$$

Note that equations (4.3) and (4.4) become equations (3.3) and (3.4) when $H_0=0$.

In analogy with our solution of model I, an integration by parts and the introduction of the variable

$$z=(w-y\,\mathrm{e}^{-\mu})/(w\,\mathrm{e}^{\nu}-y\,\mathrm{e}^{\mu+\nu}) \tag{4.5}$$

and of the function

$$G(z)=E(y)+\ln[1-z\exp(2\mu+\nu)] \tag{4.6}$$

leads to the following equation:

$$\left(1-\beta H_r\frac{(\mathrm{e}^{-\nu}-z)(z+\mathrm{e}^{\nu})}{\cosh\nu}\partial_z\right)[G(z)-(1-p)G(z\exp(-2\mu))] = p\beta H_r\left(\frac{z+\mathrm{e}^{\nu}}{\cosh\nu}-1\right)+\beta F_r+pG(z\exp(-2\mu)). \tag{4.7}$$

The location of the nearest singularity of the function $G(z)$ in the complex z plane can again be predicted by the same arguments as for model I. The cut of G runs from $-\infty$ to $-e^{\nu}$ for $H_r > 0$; from $e^{-\nu}$ to $+\infty$ for $H_r < 0$. Let us again expand $G(z)$ around $z = 0$:

$$G(z) = G(0) - \sum_{k \geq 1} a_k (pk)^{-1} \exp(k\nu)[1 - (1-p)\exp(-2k\mu)] z^k. \tag{4.8}$$

Equation (4.7) is then equivalent to

$$a_{k+1} = \sigma_k a_k + \exp(-2\nu) a_{k-1} \tag{4.9}$$

where

$$\sigma_k = 1 - \exp(-2\nu) + \frac{1+\exp(-2\nu)}{2k\beta H_r} \frac{1-\exp(-2k\mu)}{1-(1-p)\exp(-2k\mu)} \tag{4.10}$$

together with the boundary conditions $a_0 = 1$ and

$$\lim_{k\to\infty} (1/k) \ln a_k = 0 \qquad (H_r < 0) \tag{4.11a}$$

$$\lim_{k\to\infty} (1/k) \ln((-1)^k a_k) = -2\nu \qquad (H_r > 0). \tag{4.11b}$$

The free energy is related to a_1 through

$$F_r = (pH_r/\cosh \nu)(e^{\nu} a_1 - \sinh \nu). \tag{4.12}$$

4.2. *The non-frustrated case ($H_r > 0$)*

Let us first rapidly consider the case $H_r > 0$. This situation, where all magnetic fields are positive, is not very interesting from a physical point of view. Equation (4.9) is easily solved by the following continued fraction expansion:

$$a_1 = -\frac{\exp(-2\nu)}{\sigma_{1+}} \frac{\exp(-2\nu)}{\sigma_{2+}} \frac{\exp(-2\nu)}{\sigma_{3+}} \cdots. \tag{4.13}$$

The free energy is then obtained by inserting equation (4.13) into equation (4.12). At low temperature, the variables μ and ν diverge linearly with β

$$\begin{aligned} \mu &\sim \beta H_0 \\ \nu &\sim \beta(2J + H_0). \end{aligned} \tag{4.14}$$

The random part of the free energy is

$$F_r = -pH_r - 2pH_r \exp[-2(2J + H_0)\beta] + \cdots. \tag{4.15}$$

The first term is equal to the average field per site; the correction is exponentially small (just as in the non-random case) and corresponds to flipping one spin at a site for which $x_i \ll 1$.

4.3. *The frustrated case ($H_r < 0$): singular perturbation expansion*

The rest of this section is devoted to the interesting case $H_r < 0$, where the magnetic fields have both signs, and hence lead to some frustration, and to a non-trivial low-temperature limit.

In this case, the simple-minded solution (4.13) would not converge. In order to impose the behaviour (4.11a), we have to use more sophisticated methods involving Mellin transform techniques. Let $M(s)$ denote the following integral

$$M(s)=\int_C (\mathrm{d}z/2\pi\mathrm{i})z^{-s}\,\mathrm{Im}\,G'(z) \tag{4.16}$$

where the contour C encloses the cut of $G(z)$, i.e. the interval $(\mathrm{e}^{-\nu},\infty)$ with the positive orientation. It is easy to convince oneself that the function $a(s)=\exp(-\nu s)M(s)$ obeys the following difference equation

$$a(s+1)=\sigma(s)a(s)+\exp(-2\nu)a(s-1). \tag{4.17}$$

$\sigma(s)$ is given by equation (4.10) where the integer k is replaced by the complex number s. In particular $a(k)=a_k$.

We now solve equation (4.17) in the low-temperature regime, forgetting about all exponentially small terms of order $\exp(-2\nu)$. The naive low-temperature solution of equation (4.17) is

$$a^{(0)}(s)=C\prod_{n\geq 0}\sigma(n)/\sigma(s+n) \tag{4.18}$$

but it is clearly not exact, since the corresponding F_r does not vanish as p or $H_r\to 0$. It is only asymptotic to the true solution as $\mathrm{Re}\,s\to+\infty$. More precisely, if we define $b(s)$ by

$$a(s)=C\exp(-2\nu s)\prod_{n\geq 1}[\Sigma(s+n)/\Sigma(n)]b(s) \tag{4.19}$$

where $\Sigma(s)$ is the $\nu\to\infty$ limit of $\sigma(s)$

$$\Sigma(s)=1+\frac{1}{2\beta H_r s}\,\frac{1-\exp(-2\beta H_0 s)}{1-(1-p)\exp(-2\beta H_0 s)} \tag{4.20}$$

then we can drop all terms proportional to $\exp(-2\nu)$ in a consistent way and obtain

$$b(s)+b(s-1)=L(s)=\exp(2\nu s)\frac{\Sigma(0)}{\Sigma(s)}\prod_{n\geq 1}\frac{\Sigma(n)^2}{\Sigma(n+s)^2}. \tag{4.21}$$

The solution $b(s)$ giving rise to the expected analyticity properties of $G(z)$ reads

$$b(s)=\int_{0<\mathrm{Re}\,t<1}\frac{\mathrm{d}t}{2\pi\mathrm{i}}\,\frac{\pi}{\sin\pi t}\,L(s+t) \tag{4.22}$$

and hence

$$a(s)=C\int_{0<\mathrm{Re}\,t<1}\frac{\mathrm{d}t}{2\pi\mathrm{i}}\,\frac{\pi}{\sin\pi t}\exp(2\nu t)\frac{\Sigma(0)}{\Sigma(s+t)}\prod_{n\geq 1}\frac{\Sigma(n)\Sigma(s+n)}{\Sigma(s+t+n)^2} \tag{4.23}$$

where the constant C is determined by the condition $a(0)=1$ and $\Sigma(0)=1+H_0/pH_r=\bar{h}/pH_r$. It is clear from equation (4.23) that eventual poles and zeros of Σ play a crucial role in our low-temperature analysis. Poles of Σ occur only for $\mathrm{Re}\,s<0$ and will not interfere in the following. The relevant points are the zeros of Σ. This function can be written as

$$\Sigma(s)=\frac{1+2\beta H_r s}{2\beta H_r s[1-(1-p)\exp(-2\beta H_0 s)]}[1-f(\beta s)] \tag{4.24}$$

where

$$f(s)=\exp(-2H_0 s)[1+(1-p)2H_r s]/(1+2H_r s) \tag{4.25}$$

is nothing else but the average of $\exp(-2h_i s)$ over the field distribution

$$f(s)=\int \rho(x)\,\mathrm{d}x\,\exp[-2(H_0+H_r x)s]. \tag{4.26}$$

Hence the zeros of Σ are, up to a factor β, the numbers s such that $f(s)=1$. The role of complex points where the function $f(s)$ assumes the value 1 has been discussed by Derrida and Hilhorst (1983). These authors have considered the limit where $J\to+\infty$ at fixed H_0, H_r and β. Although this regime does not coincide with the low-temperature limit ($\beta\to\infty$ at fixed J, H_0, H_r), it is also governed by our equation (4.23). In fact the relation $f(s)=1$ appears in the study of other one-dimensional disordered systems, such as diffusion in a random environment. The simplest of these problems consists in finding the distribution of the random variable

$$z=1+x_1+x_1x_2+x_1x_2x_3+\dots$$

where the x_i are independent random variables with a common distribution $r(x)$. This problem has been examined by de Calan and Petritis in collaboration with ourselves (de Calan *et al* 1985) also using Mellin transform techniques.

4.4. *The frustrated case ($H_r<0$): $J\to\infty$ limit*

We now briefly discuss the behaviour of the free energy F in the Derrida–Hilhorst limit ($J\to\infty$). This behaviour crucially depends on the *sign* of the average magnetic field $\bar{h}=H_0+pH_r$. For $-pH_r<H_0$ (i.e. $\bar{h}>0$), the equation $f(s)=1$ has *one* real positive root α^*. Assume $0<\alpha^*<\beta$. Equation (4.23) yields

$$\begin{aligned}a_0&\sim C\left(\frac{\pi}{\sin\pi\alpha^*T}\exp(2\nu\alpha^*T)\frac{\Sigma(0)}{\Sigma'(\alpha^*T)}\prod_{n\geq1}\frac{\Sigma(n)^2}{\Sigma(n+\alpha^*T)^2}+1\right)\\ a_1&\sim C\Sigma(0)\end{aligned} \tag{4.27}$$

and the free energy behaves as

$$F=-J-(H_0+pH_r)-A_+\exp(-4\alpha^*J)+\dots \qquad (0<\alpha^*<\beta) \tag{4.28}$$

with

$$\begin{aligned}A_+=f'(\alpha^*)&\frac{\sin(\pi\alpha^*T)}{\pi\alpha^*T}(2\sinh\beta H_0)^{-2\alpha^*T}(1+2H_r\alpha^*)[1+2(1-p)H_r\alpha^*]\\ &\times\prod_{n\geq1}\frac{\Sigma(n+\alpha^*T)^2}{\Sigma(n)^2}.\end{aligned} \tag{4.29}$$

If $\alpha^*>\beta$, then the dominant contribution to a_0 is proportional to $\exp(2\nu)$ and

$$F=-J-(H_0+pH_r)+\mathrm{O}(\exp(-4\beta J)) \qquad (\alpha^*>\beta). \tag{4.30}$$

For $-pH_r>H_0$ (i.e. $\bar{h}<0$), the equation $f(s)=1$ has all its roots in the half-plane $\mathrm{Re}\,s<0$, and the leading one $s=\alpha^*$ is real negative. For $-\beta<\alpha^*<0$, equation (4.27) still holds and hence the free energy behaves as

$$F=-J+(H_0+pH_r)-A_-\exp(4\alpha^*J)+\ldots \qquad (-\beta<\alpha^*<0) \tag{4.31}$$

with

$$A_-=-4\frac{(pH_r+H_0)^2}{f'(\alpha^*)}\frac{\pi\alpha^*T}{\sin(\pi\alpha^*T)}\frac{(2\sinh\beta H_0)^{2\alpha^*T}}{(1+2H_r\alpha^*)[1+2(1-p)H_r\alpha^*]}$$
$$\times\prod_{n\geqslant1}\frac{\Sigma(n)^2}{\Sigma(n+\alpha^*T)^2}. \tag{4.32}$$

If $\alpha^*<-\beta$, then the dominant correction term to a_0 is of order $\exp(-2\nu)$ and

$$F=-J+(H_0+pH_r)+\mathrm{O}(\exp(-4\beta J)) \qquad (\alpha^*<-\beta). \tag{4.33}$$

The four types of low-temperature behaviour of F we have obtained in equations (4.28), (4.30), (4.31) and (4.33) agree with the general analysis of Derrida and Hilhorst (1983). Moreover we have exact expressions ((4.29) and (4.32)) for the amplitudes of the singular corrections. They admit an expansion in integer powers of T. We shall show hereafter that this remains valid for all values of J. Another interesting point which is not known for arbitrary field distributions is what happens as $J\to\infty$ when $\alpha^*=0$, i.e. when the average magnetic field vanishes. In the present case, the $\alpha^*\to0$ limit can be studied directly from equation (4.27). We obtain

$$F=-J-H_0^2\frac{2-p}{2p}\left(J-\frac{pH_0}{6(2-p)}+\frac{T}{2}\ln(2\sinh\beta H_0)-\frac{T}{2}\sum_{n\geqslant1}\frac{\Sigma'(n)}{\Sigma(n)}\right)^{-1}. \tag{4.34}$$

The exponential terms $\exp(-4|\alpha^*|J)$ are thus replaced by the simple integer power law J^{-1} for large J.

4.5. *The frustrated case ($H_r<0$): low-temperature limit*

We consider now the low-temperature behaviour of the free energy F, starting again from equation (4.23). Let us denote by α_j $(-\infty<j<+\infty)$ the non-zero complex numbers satisfying $f(\alpha_j)=1$, such that $\mathrm{Im}\,\alpha_j$ is an increasing function of the label j, α_{-j} and α_j are complex conjugates, and $\alpha_0=\alpha^*$ is the only real number of the sequence. In the particular case where the average magnetic field vanishes $(pH_r+H_0=0)$, we define $\alpha_0=0$ by continuity. When $|j|$ is large, α_j have the following asymptotic values (for $p\neq1$):

$$\alpha_j=\frac{\ln(1-p)}{2H_0}+\frac{\mathrm{i}\pi}{H_0}j+\delta\alpha_j \tag{4.35a}$$

with

$$\delta\alpha_j\sim\mathrm{i}p[4\pi(1-p)|H_rj|]^{-1} \qquad (|j|\to\infty) \tag{4.35b}$$

which are asymptotically equidistant points on a vertical straight line. The function $\Sigma(s)$ has therefore poles at $s=T\alpha_j$; this infinite string of poles now has to be considered as a whole. Equation (4.23) leads to the following expressions:

$$a_0=C\left(1+\sum_j\frac{\pi\exp[2(2J+H_0)\alpha_j]}{\sin(\pi T\alpha_j)}\frac{\Sigma(0)}{\Sigma'(T\alpha_j)}\prod_{n\geqslant1}\frac{\Sigma(n)^2}{\Sigma(n+T\alpha_j)^2}\right) \tag{4.36a}$$

$$a_1=C\Sigma(0). \tag{4.36b}$$

By eliminating C and using the relation between the functions Σ and f defined in equations (4.20) and (4.25) respectively, we obtain

$$a_1=\frac{pH_r+H_0}{pH_r}\left(1-2(pH_r+H_0)\sum_j\frac{\pi T\alpha_j}{\sin(\pi T\alpha_j)}\frac{\exp(4J\alpha_j)}{f'(\alpha_j)(1+2H_r\alpha_j)^2}\right.$$
$$\left.\times\prod_{n\geq 1}\frac{\Sigma(n)^2}{\Sigma(n+T\alpha_j)^2}\right)^{-1}. \tag{4.37}$$

This expression has a *formal* expansion in powers of T which reads

$$a_1=A_0+A_2T^2+\ldots \tag{4.38}$$

where A_2 has contributions from the sine and from the infinite product

$$A_0=\frac{pH_r+H_0}{pH_r}\left(1-2(pH_r+H_0)\sum_j\frac{\exp(4J\alpha_j)}{f'(\alpha_j)(1+2H_r\alpha_j)^2}\right)^{-1} \tag{4.39a}$$

$$A_2=\frac{\pi^2}{3}pA_0^2\sum_j\frac{\exp(4J\alpha_j)\alpha_j(1+H_r\alpha_j)}{f'(\alpha_j)(1+2H_r\alpha_j)^2}. \tag{4.39b}$$

By inserting this result in equation (4.12), we obtain the following expansion of the total free energy:

$$F=F_0+F_2T^2+\ldots \tag{4.40}$$

where

$$U_0=F_0=-J-H_0+pH_r(2A_0-1) \tag{4.41a}$$

$$\Gamma_0=\lim_{T\to 0}(C/T)=-2F_2=-4pH_rA_2 \tag{4.41b}$$

with the same notation as for model I.

Our model II therefore has generically no zero-point entropy. Its ground-state energy and specific heat amplitude are expressed in equations (4.39) and (4.41) as infinite sums over the complex numbers α_j such that $f(\alpha_j)=1$. We now examine these general expressions in a few cases of physical interest.

For small p, the sums in (4.39) are dominated by the pole at

$$\alpha^*=-(2H_r)^{-1}[1-p\exp(H_0/H_r)+O(p^2)]. \tag{4.42}$$

The physical quantities hence behave as

$$U_0=-J-H_0-pH_r\{1-2\exp[(2J+H_0)/H_r]\}+O(p^2) \tag{4.43a}$$

$$\Gamma_0=-\tfrac{1}{6}\pi^2(p/H_r)\exp[(2J+H_0)/H_r]+O(p^2). \tag{4.43b}$$

These results have the following interpretation. For small p, only a spin which feels a field $h_i<-2J$ aligns along it; the gain in energy is $(-2h_i+4J)$. If we average this quantity over the field distribution in the interval $-\infty<h_i<-2J$, we indeed obtain equation (4.43a). The specific heat amplitude corresponds to excitations at sites where $h_i\approx -2J$, and has therefore the same exponential factor.

It can easily be checked that equation (4.43) also holds in the limit of *weak disorder*: $|H_r|\ll H_0$ for all values of p, with subleading terms of order $\exp[2(2J+H_0)/H_r]$.

We consider now the *strong disorder* limit: $|H_r|\gg H_0$. In that limit, the asymptotic estimate (4.35) of the location of the poles α_j becomes exact, and the sums in equation (4.39) can be explicitly evaluated. The result for the zero-point energy is

$$U_0=-J-H_0+pH_r+2H_0[1-(1-p)^{2J/H_0}\psi(2J/H_0)]+O(H_r^{-1}) \tag{4.44}$$

where $\psi(x)$ denotes the following function:

$$\psi(x)=p^2 \sum_{-\infty<j<+\infty} \frac{\exp(2\pi \mathrm{i} jx)}{[2\pi \mathrm{i} j+\ln(1-p)]^2}. \tag{4.45}$$

$\psi(x)$ is periodic with unit period, is not differentiable at integer values of x, and has a very simple form for $0 \leqslant x \leqslant 1$:

$$\psi(x)=(1-p)^{1-x}(1-px) \qquad (0 \leqslant x \leqslant 1). \tag{4.46}$$

Equation (4.44) therefore singles out a discrete sequence of values of H_0

$$H_0=2J/N \qquad (N=1,2,3,\ldots) \tag{4.47}$$

such that the zero-temperature *magnetisation* $M_0=-\partial U_0/\partial H_0$ assumes a constant value in each interval of this sequence (for $H_r \to -\infty$), namely, on the whole interval

$$2J/(N+1)<H_0<2J/N \tag{4.48}$$

the magnetisation M_0 has the following limit:

$$M^{(N)}=-1+2(1+Np)(1-p)^{N+1} \tag{4.49}$$

as $H_r \to -\infty$.

Equations (4.43) and (4.44) show that the part of the ground-state energy due to the randomness: $U_{0r}=U_0+J+H_0$ increases linearly with $(-H_r)$ for $(-pH_r) \ll H_0$ (weak disorder regime) but decreases linearly with $(-H_r)$ at strong disorder. Figure 4 clearly shows this behaviour: U_{0r} is plotted against $(-H_r)$ for $J=1$, $H_0=1.5$ and different values of p.

Figure 5 shows plots of the zero-temperature magnetisation against H_0^{-1} for $J=1$, $p=0.4$ and different values of H_r. The genuine discontinuities of M_0 which occur at $H_r \to -\infty$ (see equation (4.49)) become steep shoulders for large finite H_r, and are still present as small ripples for $H_r=-2$. Whenever H_0 is finite, the value $H_r^{(1)}$ of H_r at

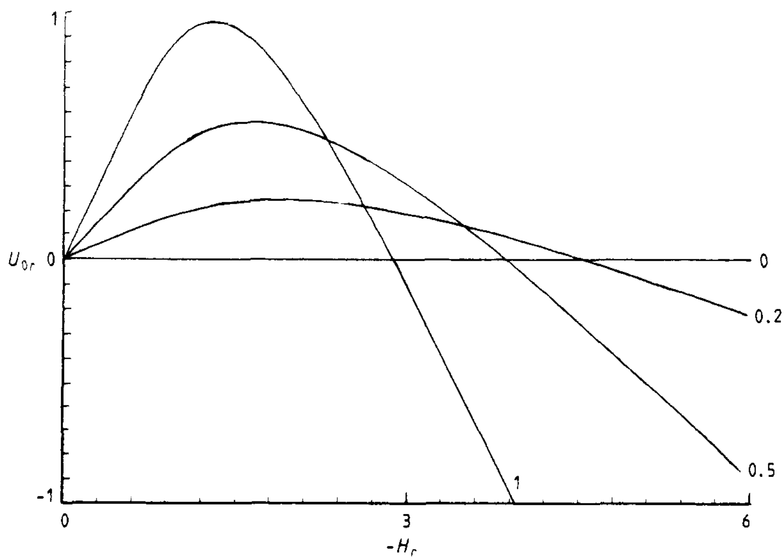

Figure 4. Plot of U_{0r}, the part due to randomness of the ground-state energy of model II with $J=1$ and $H_0=1.5$, against $(-H_r)$. Values of p are indicated on the curves.

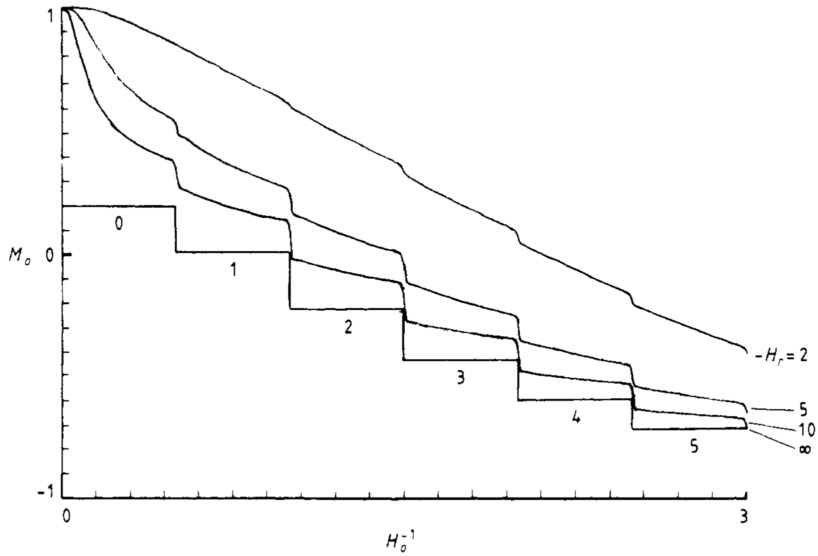

Figure 5. Plot of the zero-temperature magnetisation M_0 against H_0^{-1} for model II with $p = 0.4$ and $J = 1$. Values of H_r are indicated on the curves. The integers $0, \ldots, 5$ refer to the values of N (see equation (4.47)).

which M_0 vanishes, the value $H_r^{(2)}$ at which U_{0r} vanishes and the value $H_r^{(3)} = -H_0/p$ at which the average field $\bar{h}$ vanishes are generically all different.

The sequence of values (4.47) of H_0 also gives rise to a non-vanishing zero-point *entropy*. Indeed the sum in equation (4.39*b*) is divergent for those values of H_0, and a more correct treatment of equation (4.37), to be discussed below, shows that the expansion (4.38) contains a term $A_1 T$ with

$$A_1 = \frac{pH_0A_0^2}{4\pi^2 H_r}(1-p)^N \left[\lim_{T\to 0} \frac{1}{T} \sum_{j\neq 0} \frac{1}{j^2} \left(\frac{\pi\alpha_j T}{\sin(\pi\alpha_j T)} - 1 \right) \right]. \tag{4.50}$$

The limit inside the brackets can be evaluated by using the asymptotic estimate (4.35) of α_j and replacing the sum by an integral. The final expression for the zero-point entropy S_0 is

$$S_0 = R^2(1-p)^N \ln 2 \tag{4.51}$$

where R is related to U_{0r} through

$$R = (U_{0r} + pH_r)/2H_r. \tag{4.52}$$

This entropy is due to configurations where N successive sites have $h_i = H_0 = 2J/N$ $(i = 1, 2, \ldots, N)$ while the two adjacent spins are down $(\sigma_0 = \sigma_{N+1} = -1)$, such that the two spin configurations $\sigma_1 = \ldots = \sigma_N = +1(-1)$ are degenerate. The quantity R, which is to be interpreted as the probability that $\sigma_0 = -1$, has no simple expression in the general case. In the small p limit, equation (4.43*a*) leads to

$$R \sim p \exp[(2J + H_0)/H_r] \qquad (p \to 0) \tag{4.53}$$

which coincides with the probability for having $h_i < -2J$, as it should. In the strong disorder limit ($|H_r| \gg H_0$), equation (4.44) implies the following behaviour

$$R = p + \mathrm{O}(H_r^{-1}) \qquad (H_r \to -\infty) \tag{4.54}$$

which is also expected.

Figure 6 shows plots of the zero-temperature entropy S_0 against p for the first five values of the sequence (4.47), with $J=1$ and $H_r=-6$.

Let us show that the specific heat is linear in temperature: $C(T)=\Gamma_0 T+O(T^2)$, also when a non-trivial zero-point entropy is present. We go back to equation (4.37) and carry out carefully the necessary subtractions. Define

$$S_j=\frac{\pi\alpha_j T}{\sin(\pi\alpha_j T)} \tag{4.55a}$$

$$g_j=(1-p)^{-N}\exp(4\alpha_j J)(1+2H_r\alpha_j)^{-2}/f'(\alpha_j) \tag{4.55b}$$

$$F_j=\prod_{n\geq 1}\frac{\Sigma^2(n)}{\Sigma^2(n+\alpha_j T)}. \tag{4.55c}$$

The asymptotic behaviour of these quantities for large j reads:

$$S_j^{\mathrm{as}}=\frac{\pi jT/H_0}{\sinh(\pi jT/H_0)} \tag{4.56a}$$

$$g_j^{\mathrm{as}}=\frac{H_0}{8\pi^2 H_r^2 j^2}. \tag{4.56b}$$

We split the sum in equation (4.37) as follows:

$$\sum_j g_j S_j F_j=\sum_j g_j+\sum_j g_j^{\mathrm{as}}(S_j^{\mathrm{as}}-1)$$
$$+\sum_j[g_j(S_jF_j-1)-g_j^{\mathrm{as}}(S_j^{\mathrm{as}}-1)] \tag{4.57}$$

where the terms with $j=0$ are defined by analytic continuations when necessary. The first term contributes to the ground-state energy. The second term contributes to the zero-point entropy; as was already discussed, it can be replaced by an integral and

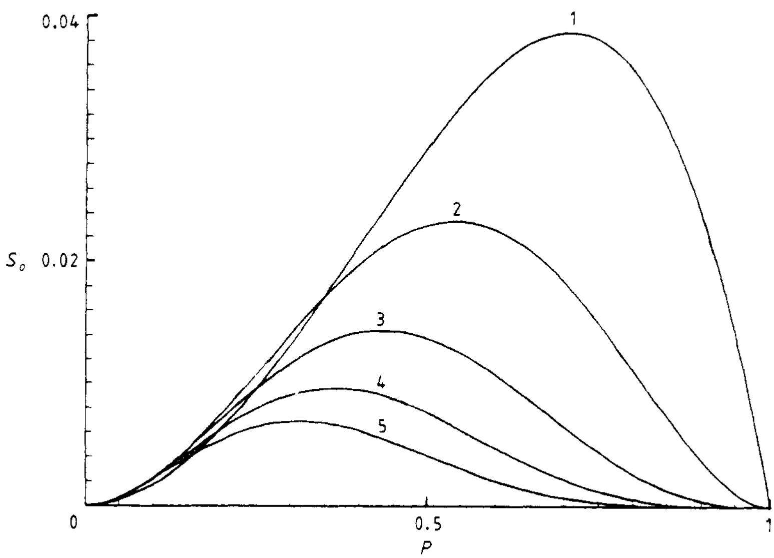

Figure 6. Plot of the zero-temperature entropy S_0 against p for model II with $J=1$ and $H_r=-6$. The integers refer to the values of N (see equation (4.47)).

leads to equations (4.51)-(4.52). The last term has a regular expansion in powers of T^2. Our final result for the specific heat amplitude Γ_0 is

$$\Gamma_0 = (-H_r)^{-1}R^3(1-p)^{2N}(\ln 2)^2 - \tfrac{1}{6}\pi^2 H_0^{-1}R^2(1-p)^N \times \sum_j [1+8H_0H_rg_j\alpha_j(1+\alpha_jH_r)]. \tag{4.58}$$

This method can be extended to compute the higher-order terms in the low-T expansion (4.40) of the free energy, both when S_0 is vanishing and non-vanishing. Our expectation is that, just as in model I, the free energy has an asymptotic expansion in integer powers of T.

4.6. *The frustrated case ($H_r < 0$): convergent algorithm at finite temperature*

We finally mention an efficient way of finding numerical values of the free energy for all finite temperatures. Contrary to model I, the three-term recurrence relation (4.9) together with its boundary condition (4.11*a*) cannot be treated numerically as it stands. A convergent algorithm can be found by expanding the function $G(z)$ around the point $z=-e^{\nu}$. Define a function $K(t)$ such that

$$K(t) = G(z)|_{z=-e^{\nu}+2t\cosh\nu}. \tag{4.59}$$

Equation (4.7) is then equivalent to

$$[1-2\beta H_r t(1-t)\partial_t]K(t) = \beta F_r + p\beta H_r(2t-1) + [1-2(1-p)\beta H_r t(1-t)\partial_t] \times K[\exp(-2\mu)t + (1-\exp(-2\mu))/(1+\exp(-2\nu))]. \tag{4.60}$$

If we expand K in a power series in t

$$K(t) = K(0) - p\sum_{n\geq 1}(K_n/n)t^n \tag{4.61}$$

and define quantities L_n through $L_0 = 0$ and

$$L_n = n\exp(-2n\mu)\sum_{l\geq n}\frac{K_l}{l}\binom{l}{n}\left(\frac{1-\exp(-2\mu)}{1+\exp(-2\nu)}\right)^{l-n} \qquad (n\geq 1) \tag{4.62}$$

then the K_n have to obey the following equation:

$$n^{-1}K_n + 2\beta H_r(K_{n-1}-K_n) = n^{-1}L_n + 2(1-p)\beta H_r(L_{n-1}-L_n) \tag{4.63}$$

together with the boundary condition $K_0 = 1$.

The free energy is then given by

$$F_r = pH_r + pT\sum_{n\geq 1}(K_n/n)[(1-\exp(-2\mu))/(1+\exp(-2\nu))]^n. \tag{4.64}$$

Equation (4.63) involves K_{n-1} in the LHS, and in the RHS through L_{n-1}; it can therefore be used to express K_{n-1} as a function of K_p with $p \geq n$. Since K_n decrease for large n as

$$K_n \sim n^{1/2\beta H_r} \tag{4.65}$$

the following numerical algorithm for the free energy is efficient: take K_n equal to their asymptotic estimate (4.65) for $n > n_0$, where n_0 is some large number; then use equation (4.63) to determine K_{n_0}, K_{n_0-1}, etc, down to K_0, and multiply all the K_n by a scale factor such that the condition $K_0 = 1$ is satisfied; the free energy is then given by (4.64).

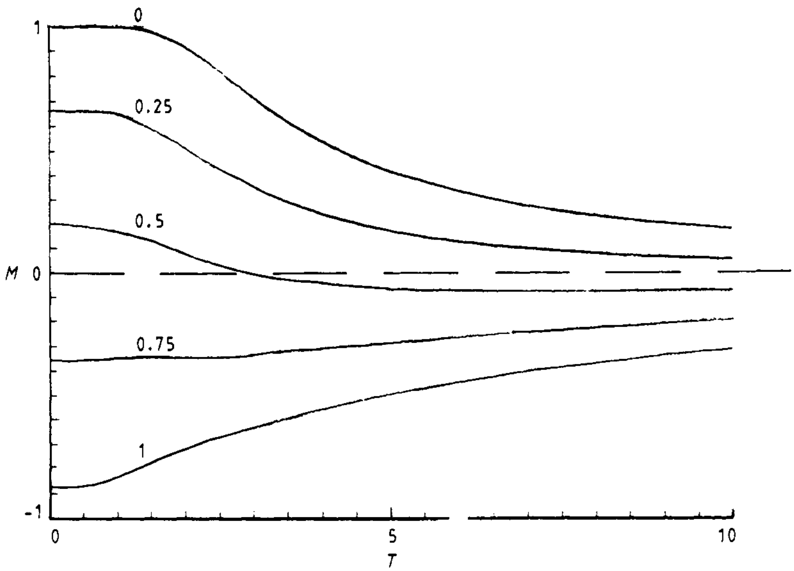

Figure 7. Plot of the magnetisation of model II with $J=1$, $H_0=1.5$ and $H_r=-5$ against temperature. Values of p are indicated on the curves.

We have computed by this method the dependence of the magnetisation M on temperature. Figure 7 shows plots of M against T for different values of p ($p=0$, 0.25, 0.5, 0.75 and 1). The parameters read $J=1$, $H_0=1.5$, $H_r=-5$. These curves are very smooth for generic values of the parameters. Note, however, that M may *change sign* as temperature is varied: this occurs for $p=0.5$ in the present case.

5. Conclusions

In this paper, we have presented exact solutions of two particular random field ferromagnetic Ising models in one dimension.

For model I (symmetric exponential distribution without an external field), we give an explicit continued fraction expansion of the free energy at finite temperature. The study of the low-temperature limit needs a more refined analysis of an effective differential equation. The result is that the free energy can be expanded in integer powers of temperature. The zero-point entropy is non-vanishing for all values of the parameters if $p \neq 1$ (diluted randomness).

Model II (non-symmetric exponential distribution) may be considered as follows: a fraction p of the sites has a negative random field $H_r x_i$. If the external field H_0 is negative, then the ground state remains the ferromagnetic one, since there is no frustration, and we find the simple solution (4.12)-(4.13). (This expression was in fact evaluated for positive H_0 and H_r, but the problem is clearly invariant under changes of sign of all fields.) If a positive external field H_0 is applied, some frustration is introduced in the system. The frustrated case is physically more interesting and technically more difficult. The large exchange coupling limit and the low-temperature limit are studied through a singular perturbation expansion of our recurrence relations. The magnetisation may be non-zero when the average magnetic field vanishes, even at zero temperature. Moreover, it may change sign when temperature is varied.

For a discrete set of values of the steady part H_0 of the random fields, the zero-temperature entropy is non-vanishing and the zero-temperature magnetisation is discontinuous in the strong disorder limit.

In both models the specific heat behaves linearly in temperature at low temperature, whenever some frustration is present, irrespective of the vanishing or non-vanishing of the zero-point entropy. This behaviour, which seems to be quite general, is very different from the large J limit, where the free energy exhibits a continuously varying critical exponent (Derrida and Hilhorst 1983). We have verified this prediction explicitly, calculated the (temperature-dependent) amplitude of that singular part, and also obtained the result for the limiting case where α^* vanishes.

The free energy of both models has an (asymptotic) expansion in *integer* powers of temperature. This fact seems to be very general.

Let us finally mention that the antiferromagnetic version of model II is also soluble by our method (model I is invariant under the change $J \rightarrow -J$). Our methods of analysis of the low-temperature singularities (effective differential equation for model I, and singular perturbation expansion of the Mellin transform for model II) are also applicable to other singularities of these models, such as Lee-Yang singularities occurring for complex values of H_0 and H_r.

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